Math 206A Lecture 24 Notes

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November 26, 2018

1 Perles' Theorem and Point and Line Configurations

1.1 Perles' theorem

Last time, we dealt with Steinitz's theorem, which we pretended to prove. Here is a corollary.

Corollary 1.1. *Let $P \subseteq \mathbb{R}^3$ be a convex polytope. Then there exists a $P' \subseteq \mathbb{Q}^3$ such that $\alpha(P) \simeq \alpha(P')$.*

This comes from the proof of Steinitz's theorem, not the theorem itself. The proof actually constructs a rational polytope.

Corollary 1.2. *Let $P \subseteq \mathbb{R}^3$ be a convex polytope. Then there exists a $P' \subseteq \mathbb{Q}^3$ such that $\|P - P'\| < \varepsilon$ and $\alpha(P) \simeq \alpha(P')$.*

Here the norm is the maximum distance between corresponding vertices of the polytopes.

Proof. If the polytope is simplicial, we can just perturb each vertex by some ε to make it rational.

What is nonobvious is that for all P, there exists a sequence of vertex-face perturbations with final polytope $P' \subseteq \mathbb{Q}^3$. $\square$

Here is a conjecture: The first corollary generalizes to all $P \subseteq \mathbb{R}^d$ for $d \geq 4$. It is wrong, however.

Theorem 1.1 (Perles[1], 1960s)**.** *There exist $P \subseteq \mathbb{R}^d$ such that for all $P' \in \mathbb{Q}^d$, $\alpha(P) \not\simeq \alpha(P')$.*

How can we prove this?

[1] Perles was the advisor of Gil Kalai.

1.2 Point and line configurations

Definition 1.1. A **point and line configuration** $K = (V, L)$ is a set of "points" $V = \{v_1, \ldots, v_n\}$ and "lines" $L = \{\ell_1, \ldots, \ell_m\}$, where $\ell_i \subseteq 2^V$.

This is an abstract set-theoretic object, like a graph.[2]

Definition 1.2. A **realization** of K in $\mathbb{F}^2$ is a map $f : V \to \mathbb{F}^2$ and a map $\tilde{f} : L \to \{\text{lines in } \mathbb{F}^2\}$ such that $v_i \in \ell_j$ iff $f(v_i) \in \tilde{f}(\ell_j)$.

Example 1.1. A **Fano plane** is a (triangular) configuration with $V = \{1, 2, 3, 4, 5, 6, 7\}$.[3] There exists a realization over $\mathbb{F}_2$ but not over $\mathbb{R}$.

Example 1.2. The **Pappus configuration** looks like a graph of $K_{3,3}$ with vertices at the intersection points of the edges and two lines connecting the 3 vertices on each side.

Theorem 1.2 (Pappus)**.** *There does not exist a realizatio nof the Pappus configuration over* $\mathbb{R}$.

Proof. The idea is that these middle three vertices are always colinear, but there is no line containing them specified in the configuration. □

Example 1.3. Another example is the **Desargues configuration**.[4]

Theorem 1.3 (Desargues)**.** *The Desargues configuration cannot be realized over* $\mathbb{R}$.

Theorem 1.4. *There exists a configuration* $K = (V, L)$ *that is realizable over* $\mathbb{R}$ *but not over* $\mathbb{Q}$.

Remark 1.1. In fact, there exists a configuration which is realizable over the algebraic numbers $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$.

Proof. The proof is heavily pictoral, so you'll have to read about it in Professor Pak's book. The idea is universality theorems. Basically, we can encode algebraic equations using point and line-configurations. Construct an algebraic equation which does not have solutions over $\mathbb{Q}$. □

[2] Hilbert was interested in these as a possible foundation for geometry.
[3] Look up a picture online!
[4] I really can't draw this, so look it up online.